

Integral inference method for $da/dN-\Delta K$ and $P-da/dN-\Delta K$ curves

FU Hui-min, WU Qiong

(Research Center of Small Sample Technology,
Beijing University of Aeronautics and Astronautics, Beijing 100191, China)

Abstract: The integral inference method for fatigue crack growth (FCG) curve was established based on the two-variance regression theory. The estimates of $da/dN-\Delta K$ curve and $P-da/dN-\Delta K$ curve with high confidence level and high survival probability were also given. FCG data scatter with stress intensity factor ranges in reality, which limits the group test method's application. The presented method analyzes FCG curve as an integration, thus it can exploit the lengthwise information among the FCG rates corresponding to different stress intensity factor ranges. Therefore, compared with group test method, the presented method can not only save specimens, but also have higher precision in the evaluations of $da/dN-\Delta K$ and $P-da/dN-\Delta K$ curves.

Key words: aero-engine; fatigue crack growth (FCG); crack growth rate; integral estimate; reliability

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Introduction

The fatigue crack growth (FCG) rate da/dN is an important basis for damage tolerance design and is also a major material index to characterize the fatigue and fracture properties^[1-4]. The FCG curve with high confidence level and high survival probability (namely reliability), i. e. $P-da/dN-\Delta K$ curve, is a core performance parameter for the design and life determination of aero-engine, especially for its turbine disk, main shaft, case, etc^[1,5]. Currently, $P-da/dN-\Delta K$ curve is mainly obtained by the group test method.

The group test method is an ensemble average technique essentially^[6], which involves two main stages. In the first stage, conduct FCG tests with a lot of specimens to collect sufficient amount of FCG data ($da/dN, \Delta K$), and estimate the percentile of FCG rate da/dN and its confidence limit for each stress intensity factor

range ΔK . Then in the second stage, fit the estimates of percentiles as well as its confidence limits as functions of stress intensity factor range. The $P-da/dN-\Delta K$ curve with confidence level and survival probability of interest can thus be obtained. However, this method has some apparent drawbacks which limit its application in engineering. Firstly, this method requires recording the numbers of load cycle to reach specified crack lengths. But due to the uncontrollability of crack length, it is difficult to acquire the exact cycle number to reach a specified crack length with the current experimental equipments. Therefore, the data set ($da/dN, \Delta K$) are scattering points in reality. Though one can resort to interpolation/extrapolation strategy to overcome this problem, it inevitably introduces unnegligible errors. Secondly, a large amount of specimens are required in this method, which is time-consuming, costly and always needs a lot of manpower. Obviously, these are difficult to be

satisfied in many practical situations^[7-8]. Thirdly, the two-stage statistical analysis might lose some information of the parameters and lead to accumulation of computational errors, which usually results in insufficient estimates. Additionally, very less crack growth mechanisms are taken into consideration in this method, which have been criticized by other researchers^[9].

In this paper, the integral inference method for FCG curve was established, according to the two-variance regression theory presented in Ref. [10]. The estimates of $da/dN-\Delta K$ curve and $P-da/dN-\Delta K$ curve with high confidence level and high survival probability were also given. In this method, the FCG curve is treated as an integration and the lengthwise information among the FCG rates corresponding to different stress intensity factor ranges are exploited. Hence, compared with group test method, this method can evaluate the $da/dN-\Delta K$ and $P-da/dN-\Delta K$ curves with higher precision and less specimens.

1 Fundamental model for fatigue crack propagation

If the experimental conditions, such as environment and stress ratio, are prespecified, the typical FCG curve is sigmoidal in shape and comprises three regions. The first region is the near-threshold region, in which very slow crack growth occurs and no growth occurs below the threshold stress intensity factor range. The second one is the stable region, where the fatigue crack propagates with moderate velocities. And the third region is the higher growth rate portion of the FCG curve, in which region rapid and unstable crack growth occurs as final fracture is approached.

Among these regions, the stable region accounts for a large share of the fatigue crack propagation life. The near-threshold region is degenerated from the second region to some extent and the unstable region only takes a little proportion (about 2% ~ 5%) of the crack propagation life^[3-4]. Therefore, from the viewpoint

of fatigue life, the stable region is the most important portion and the current research mainly focuses on the study of it.

1.1 Mean value model of log crack growth rate

Many empirical and phenomenological-based crack growth models have been proposed in the literature^[1-4], among which the most well known and most widely used one is the Paris-Erdogan law which well describes the relationship between FCG rate and stress intensity factor range in the stable region. Paris-Erdogan law can be expressed as

$$\frac{da}{dN} = C(\Delta K)^\alpha \quad (1)$$

where C and α are material constants.

Take the common logarithm on both sides of Eq. (1), we have

$$\lg \frac{da}{dN} = \lg C + \alpha \lg(\Delta K) \quad (2)$$

Then denote

$$\begin{cases} y = \lg \frac{da}{dN} \\ x = \lg(\Delta K) \\ a = \lg C \\ b = \alpha \end{cases} \quad (3)$$

Substituting it into Eq. (2), we can obtain

$$y = a + bx \quad (4)$$

Obviously, the FCG curve in the stable region follows a straight line in log-log scale, which provides a good approximation of the majority of experimental data.

1.2 Scatter of log crack growth rate

A large number of FCG data have been analyzed in this investigation. And it is revealed that FCG data essentially involves two kinds of scatter, one of which is caused by intrinsic sources such as variability in material, machining and manufacturing, and the other is caused by extrinsic sources such as variability in loading conditions, environmental conditions and errors of measurement.

The intrinsic scatter can be represented by a totally correlated random variable, which does

not vary during the crack propagation process. While the extrinsic scatter can be described by an independent random variable with a constant variance, which is totally uncorrelated at different test times.

1.3 Two-variance regression model for crack propagation

In the literature, large efforts have been placed on the trying to determine the statistical distribution of FCG rate da/dN , and it has been indicated that da/dN follows a lognormal distribution generally^[1-4]. Hence, the logarithm of FCG rate follows normal distribution.

Therefore, the FCG curve can be well represented by the two-variance regression model proposed in Ref. [10]. According to Section 1.2 and Section 1.3, the FCG curve can be represented by

$$y = a + bx + \epsilon + e \quad (5)$$

$$\epsilon \sim N(0, \sigma_\epsilon^2), \quad e \sim N(0, \sigma_e^2) \quad (6)$$

where $a, b, \sigma_\epsilon \geq 0$ and $\sigma_e \geq 0$ are unknown parameters to be estimated, $\epsilon = \epsilon(x)$ is a random variable, which is totally correlated for different x , and $e = e(x)$ is uncorrelated variable independent of each other for different x . Here, σ_ϵ^2 and σ_e^2 are called correlated variance and independent variance, respectively.

2 Two-variance regression analysis method for da/dN - ΔK curve

Suppose m specimens are put in FCG tests. And n_i FCG data pairs $((da/dN)_{ij}, \Delta K_{ij})$ for the i th specimen can be obtained from crack length a and cycle number N by the secant method or the incremental polynomial method. Subsequently, transforming the FCG rate data pairs in terms of Eq. (3) yields n data pairs (x_{ij}, y_{ij}) . Here, $i = 1, 2, \dots, m; j = 1, 2, \dots, n_i$; and $n = \sum_{i=1}^m n_i$ which is the total number of obtained FCG data pairs for all the m specimens.

From the two-variance regression analysis method^[10], we can acquire estimates of the model parameters as follows:

$$\hat{a} = \bar{y} - \hat{b} \bar{x} \quad (7)$$

$$\hat{b} = \frac{l_{xy}}{l_{xx}} \quad (8)$$

$$E(\theta) = (1 - \theta) \left(2 \frac{l_{xy}}{l_{xx}} l_{xy\theta} - \frac{l_{xy}^2}{l_{xx}^2} l_{xx\theta} - l_{yy\theta} \right) - \frac{n^* - 2}{n - 2} \left(l_{yy} - \frac{l_{xy}^2}{l_{xx}} \right) = 0 \quad (9)$$

and

$$\hat{\sigma}_\epsilon^2 = \frac{\theta}{(n - 2)(1 - \theta)} \left(l_{yy} - \frac{l_{xy}^2}{l_{xx}} \right) \quad (10)$$

$$\hat{\sigma}_e^2 = \frac{1}{n - 2} \left(l_{yy} - \frac{l_{xy}^2}{l_{xx}} \right) \quad (11)$$

for $0 \leq \theta < 1$, or

$$\hat{\sigma}_\epsilon^2 = \frac{1}{m - 1} \left(2 \frac{l_{xy}}{l_{xx}} l_{xy\theta} - \frac{l_{xy}^2}{l_{xx}^2} l_{xx\theta} - l_{yy\theta} \right) \quad (12)$$

for $\theta = 1$, where

$$\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \quad (14)$$

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \quad (15)$$

$$\bar{x} = \frac{1}{n^*} \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{x_{ij}}{I_i(\theta)} \quad (16)$$

$$\bar{y} = \frac{1}{n^*} \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{y_{ij}}{I_i(\theta)} \quad (17)$$

$$n^* = \sum_{i=1}^m \frac{n_i}{I_i(\theta)} \quad (18)$$

$$I_i(\theta) = 1 + (n_i - 1)\theta \quad (19)$$

$$l_{xx} = \theta \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i (x_{ij} - \bar{x}_i)^2}{I_i(\theta)} + (1 - \theta) \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{(x_{ij} - \bar{x})^2}{I_i(\theta)} \quad (20)$$

$$l_{yy} = \theta \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i (y_{ij} - \bar{y}_i)^2}{I_i(\theta)} + (1 - \theta) \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{(y_{ij} - \bar{y})^2}{I_i(\theta)} \quad (21)$$

$$l_{xy} = \theta \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i (x_{ij} - \bar{x}_i)(y_{ij} - \bar{y}_i)}{I_i(\theta)} + (1 - \theta) \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{(x_{ij} - \bar{x})(y_{ij} - \bar{y})}{I_i(\theta)} \quad (22)$$

$$l_{xx\theta} = - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i^2 (\bar{x}_i - \bar{x})^2}{I_i^2(\theta)} \quad (23)$$

$$l_{yy\theta} = - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i^2 (\bar{y}_i - \bar{y})^2}{I_i^2(\theta)} \quad (24)$$

$$l_{xy\theta} = - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{n_i^2 (\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y})}{I_i^2(\theta)} \quad (25)$$

The estimate of θ can be obtained by solving Eq. (9). According to Ref. [10], Eq. (9) can be solved by a binary search. Then substitute the estimate of θ into Eq. (7) and Eq. (8), we can obtain the estimates $\hat{a}$ and $\hat{b}$.

Therefore, the regression equation of da/dN - ΔK curve in log-log scale is

$$\hat{y} = \hat{a} + \hat{b}x \quad (26)$$

Then, from Eq. (3), we have FCG curve as follows

$$\frac{da}{dN} = 10^{\hat{a}} (\Delta K)^{\hat{b}} \quad (27)$$

3 Two-variance regression analysis method for P - da/dN - ΔK curve

From Ref. [10], the P - da/dN - ΔK curve in log-log scale can be approximately given by

$$y_R = a_R + \hat{b}x \quad (28)$$

where $y_R = \lg\left(\frac{da}{dN}\right)_R$, $\left(\frac{da}{dN}\right)_R$ is the percentile value of FCG rate corresponding to an arbitrary survival probability (i. e. reliability) R , and

$$a_R = \hat{a} + u_{Rc_0} \hat{\sigma} + u_{\gamma c_0} \hat{\sigma} \sqrt{\frac{1}{n^*} \left(1 - \frac{u_{\gamma}^2}{w}\right) + \frac{u_R^2}{w}} \quad (29)$$

$$\hat{\sigma}^2 = \hat{\sigma}_{\epsilon}^2 + \hat{\sigma}_e^2 \quad (30)$$

$$c_0 = \beta \left(1 - \frac{u_{\gamma}^2}{w}\right)^{-1} \quad (31)$$

$$\beta = \sqrt{\frac{\nu}{2}} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} \quad (32)$$

here, u_R and u_{γ} are standard normal deviates corresponding to survival probability R and confidence level γ , respectively; $\nu = \text{Int}(n^* + \theta - 2)$,

in which $\text{Int}(\cdot)$ is the greatest integer function; and $\Gamma(\cdot)$ is the gamma function. For $\nu \geq 5$, β can

be approximated by $\beta \approx \sqrt{\frac{2\nu+1}{2\nu}}$.

Furthermore, the P - da/dN - ΔK curve with confidence level γ and survival probability R is

$$\left(\frac{da}{dN}\right)_R = 10^{a_R} (\Delta K)^{\hat{b}} \quad (33)$$

where a_R and $\hat{b}$ are parameters given in Eq. (29) and Eq. (8), respectively.

4 Applications

Table 1 gives the FCG data of Q235A steel obtained from constant amplitude fatigue tests^[11]. Three compact tension (CT) specimens are subjected to cyclic loads with a load ratio of 0.1 and maximum loads of 19.6, 17.6 kN and 14.7 kN, respectively. It can be seen that the FCG rates da/dN are scattering with the stress intensity factor ranges ΔK , thus we can not obtain P - da/dN - ΔK curve from the group test method.

According to the two-variance regression method presented in Section 2, the regression equation is given by

$$\hat{y} = -5.1871 + 3.5143x$$

Thus the da/dN - ΔK curve of Q235A steel is

$$\frac{da}{dN} = 6.5002 \times 10^{-6} (\Delta K)^{3.5413}$$

Then, from Eq. (28), we can acquire the P - da/dN - ΔK curve in log-log scale, with confidence level of 95% and survival probability of 99.9%

$$y_R = -4.7423 + 3.5143x$$

Therefore, we have

$$\left(\frac{da}{dN}\right)_R = 1.8103 \times 10^{-5} (\Delta K)^{3.5413}$$

which is the P - da/dN - ΔK curve of Q235A steel. Fig. 1 shows the obtained da/dN - ΔK and P - da/dN - ΔK curves of Q235A steel. It can be seen that the presented method has a reasonable accuracy for engineering application.

Table 1 Fatigue crack growth rate data for Q235A steel

Specimen 1		Specimen 2		Specimen 3	
$\Delta K /$ (MPa $\cdot$ mm $^{0.5}$)	(da/dN)/ 10^{-4} (mm/cycle)	$\Delta K /$ (MPa $\cdot$ mm $^{0.5}$)	(da/dN)/ 10^{-4} (mm/cycle)	$\Delta K /$ (MPa $\cdot$ mm $^{0.5}$)	(da/dN)/ 10^{-4} (mm/cycle)
26.53	0.567	22.38	0.322	19.13	0.120
27.93	0.662	23.56	0.609	20.09	0.349
28.06	0.886	24.83	0.476	21.14	0.210
30.54	1.103	25.88	0.714	22.07	0.321
32.09	1.084	27.03	0.666	23.10	0.360
33.54	1.390	28.24	1.189	23.90	0.473
35.19	1.803	29.70	0.852	25.33	0.532
37.19	1.790	31.43	1.219	25.73	0.688
39.06	2.239	32.77	1.881	28.08	0.782
41.04	2.411	34.52	2.062	29.45	0.976
43.25	3.776	36.33	1.564	31.06	1.542
45.63	3.513	38.16	4.620	32.74	1.186
48.24	5.467	40.42	3.546	31.72	2.141
51.49	5.189	42.16	3.521	36.89	2.116
54.96	10.00	45.88	4.242	39.49	2.660
58.56	6.806	48.98	5.969	42.22	3.568
62.62	16.67	52.30	6.845	44.92	4.306
67.55	16.98	55.99	8.557	48.14	5.029
73.32	17.78	60.70	11.63	52.51	7.105
78.90	18.48	65.16	15.26	56.42	7.421
85.59	55.67	70.53	25.30	61.07	11.84
		77.50	28.40	67.36	13.08
		86.16	77.03	74.65	22.00

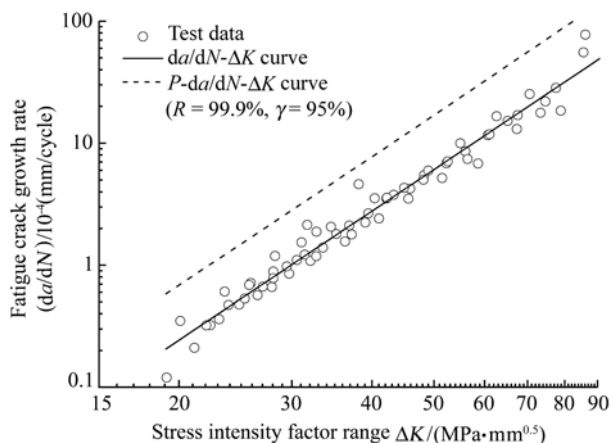


Fig. 1 Fatigue crack growth rate curves of Q235A steel

5 Conclusions

The main conclusions of this paper are as follows:

- 1) Based on the investigation of a large amount of FCG data, it is revealed that the FCG data involves two kinds of scatter, one of which is caused by intrinsic sources and the other is caused by extrinsic sources. They can be represented by totally correlated random variable and independent random variable, respectively.
- 2) The analysis of the mean value and scat-

ter of the log FCG rate shows that the FCG curve can be well described by the two-variance regression model presented in Ref. [10].

3) The regression equation and confidence limit curves with high confidence level and high survival probability of FCG data are given in this paper.

4) The presented method exploits the lengthwise information in the FCG rates corresponding to different stress intensity factor ranges. Thus, it not only has higher precision but also saves specimens, compared with group test method.

5) There are extensive FCG data in engineering handbooks^[11-12], which are mainly scattering data points. From these data, we can obtain P - da/dN - ΔK curve with desired confidence level and survival probability, which is very valuable in practical application, especially for the design and life determination of aero-engine.

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